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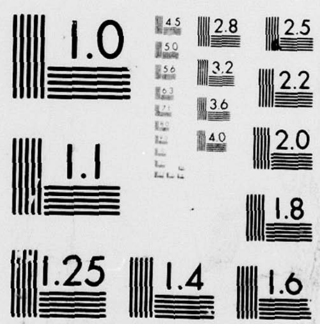

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CONVERGENCE OF ABSTRACT SPLINES

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CONVERGENCE OF ABSTRACT SPLINES

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ABSTRACT

Shekhtman [3] gives a sufficient condition for the convergence of abstract splines. We show that his condition is not necessary but that a slight perturbation of his condition is both necessary and sufficient. In the process, we also give a necessary and sufficient condition for a sequence of abstract spline projectors to be bounded.

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SIGNIFICANCE AND EXPLANATION

Various generalizations of the polynomial spline have been given over the years in an attempt to understand the mathematical mechanisms on which the spline notion is based. This led to the notion of a spline as the smoothest interpolant to some given data, with both "smoothness" and "interpolation" taken in a rather simple, but abstract sense. Recently, Shekhtman [3] showed that the resulting spline interpolant converges to the function from which the data are taken, under rather reasonable conditions on the "interpolation" notion concept used. But he left open the question whether these conditions are necessary, and, as it turns out, made an unnecessarily strong assumption concerning the "smoothness" notion used.

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CONVERGENCE OF ABSTRACT SPLINES

Carl de Boor

Shekhtman [3] gives a sufficient condition for the convergence of abstract splines. We show that a slight perturbation of his condition is both necessary and sufficient. In the process, we also give a necessary and sufficient condition for a sequence of abstract spline projectors to be bounded.

It seems most convenient to discuss the abstract spline (as introduced by Atteia [1]) in the following way. Let X be a Hilbert space, and let Λ be a set of continuous linear functionals on X. Among the possibly many elements of X which agree with a given $x \in X$ on Λ, i.e., from the flat

$$x + \ker \Lambda ,$$

we attempt to select a particular one by the prescription that it should minimize $\|Ty\|$ over y in $x + \ker \Lambda$. Here,

$$\ker \Lambda := \Lambda_\perp := \bigcap_{\lambda \in \Lambda} \ker \lambda$$

and T is a given bounded linear map on X to some Hilbert space Z. We assume that

$$\begin{array}{ll} & \ker T \cap \ker \Lambda = \{0\} \\ (1) & \text{ran } T \text{ is closed} \\ & \dim \ker T < \infty . \end{array}$$

This insures that the minimization problem has one and only one solution, and this

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solution is the abstract spline, or, more precisely, the (T,Λ)-spline interpolant to x in question. We shall denote it by

$$px \, .$$

It is obvious that the map p so defined is a linear projector on X, with

$$\ker p = \ker \Lambda \, .$$

Further, the minimization problem and its solution do not change if we replace Λ by its closed linear hull, i.e., by $(\Lambda_\perp)^\perp = (\Lambda\perp)\perp$. We therefore assume from now on that

$$\Lambda \text{ is a closed linear subspace of } X^* \ (= X) \, .$$

Remark: Here, we follow Shekhtman [3] in assuming that $\dim \ker T < \infty$ (which is essential for his proofs). Actually, Atteia [1] and others do not make this assumption, but prove existence of px under the weaker assumption that $(\ker T) + (\ker \Lambda)$ is closed.

Let now (Λ_n) be a given sequence of closed subspaces of $X^* = X$ satisfying

(2) $$\ker T \cap \ker \Lambda_n = \{0\} \, , \quad \text{all } n \, .$$

Then Shekhtman is concerned with the question of when the corresponding sequence (p_n) of spline projectors converges pointwise, or strongly, to 1. In this connection, the following well known lemma is a consequence of the uniform boundedness principle and Lebesgue's Inequality

$$\|x - p_n x\| \leq \|1 - p_n\| \operatorname{dist}(x, \operatorname{ran} p_n) \, .$$

Lemma 1. $p_n \xrightarrow{s} 1$ iff (p_n) is bounded and $\underline{\lim}_{n\to\infty} \operatorname{ran} p_n = X$.

Here, we use the abbreviation

(3) $$\underline{\lim}\, A_n := \{\lim a_n : a_n \in A_n, \text{ all } n\}$$

with $\lim a_n$ taken in the norm on X.

Unfortunately, the spline interpolation projection is given in terms of T and (Λ_n) and the character of $\operatorname{ran} p_n$ is, in general, not known a priori. It is therefore important to give conditions for the convergence of p_n in terms of T and (Λ_n).

Theorem 1. (Shekhtman [3]) If $\underline{\lim}\, \Lambda_n = X$, then $p_n \xrightarrow{s} 1$.

The major part of the proof is spent in proving

Lemma 2. If $\underline{\lim}\, \Lambda_n = X$, then (p_n) is bounded.

I want to give a different proof of this lemma by first proving

Proposition 1. (p_n) is bounded iff

$$\kappa_\infty := \sup_n \sup \left\{\frac{(x,y)}{\|x\|\|y\|} : x \in \ker T,\ y \in \ker \Lambda_n\right\} < 1 . \tag{4}$$

In effect, (4) is a quantitative strengthening of (2) since it says that the *inclination between* $\ker T$ *and* $\ker \Lambda_n$ should be bounded away from 1. Here, the inclination between two subspaces A and B is, by definition, the cosine of the smallest angle between them, i.e., the number

$$\operatorname{incl}(A,B) := \sup_{a \in A,\, b \in B} \frac{(a,b)}{\|a\|\|b\|} = \|P_A|_B\| = \|P_B|_A\| , \tag{5}$$

with P_A, P_B the orthogonal projector onto A and B, respectively.

Proof of Proposition 1. It is sufficient to prove that, for the (T,Λ)-spline projector p,

$$(1-\kappa^2)^{-1/2} \le \|p\| \le 1 + \|(T|_{\operatorname{ran} Q})^{-1}\|(1-\kappa^2)^{-1/2} \tag{6}$$

with Q the orthogonal projector onto $(\ker T)^\perp$ and

$$\kappa := \operatorname{incl}(\ker T, \ker \Lambda) .$$

For the lower bound, let $P := P_\Lambda$ so that $\ker P = \ker \Lambda = \ker p$. Since $px = x$ for x in $\ker T$, we have

$$\|x\| = \|px\| \leq \|p\| \operatorname{dist}(x, \ker p), \quad \text{for all } x \in \ker T,$$

while $\operatorname{dist}(x, \ker p) = \operatorname{dist}(x, \ker P) = \|Px\|$. Consequently,

$$\|p\| \geq \sup_{x \in \ker T} \frac{\|x\|}{\|Px\|},$$

while

$$\inf_{x \in \ker T} \left(\frac{\|Px\|}{\|x\|}\right)^2 = 1 - \sup_{x \in \ker T} \frac{\|(1-P)x\|^2}{\|x\|^2} = 1 - \|(1-P)|_{\ker T}\|^2 = 1 - \kappa^2$$

using (5) and the fact that $1 - P = P_{\ker \Lambda}$.

For the upper bound, recall from Golomb [2; (3.8)] (or else verify directly) that

$$p = 1 - T_0^{-1}(P_{T[\ker \Lambda]})T \tag{7}$$

with $T_0 := T|_{\ker \Lambda}$. Consequently,

$$\|p\| \leq 1 + \|T_0^{-1}\| \|T\|,$$

and we calculate $\|T_0^{-1}\|$ as

$$\|T_0^{-1}\| = \sup_{x \in \ker \Lambda} \|x\| / \|Tx\|.$$

But, since $Tx = TQx$ (using the orthoprojector Q onto $(\ker T)\perp$ introduced earlier), we have

$$\|x\| / \|Tx\| = \frac{\|x\|}{\|Qx\|} \frac{\|Qx\|}{\|TQx\|}$$

hence

$$\|T_0^{-1}\| \leq \sup_{x \in \ker \Lambda} (\|x\| / \|Qx\|) \|(T|_{\operatorname{ran} Q})^{-1}\|$$

while, as before, $\inf\limits_{x \in \ker \Lambda} \frac{\|Qx\|^2}{\|x\|^2} = 1 - \sup\limits_{x \in \ker \Lambda} \frac{\|(1-Q)x\|^2}{\|x\|^2} = 1 - \|(1-Q)|_{\ker \Lambda}\|^2 = 1 - \kappa^2$

by (5) and since $1 - Q = P_{\ker T}$. |||

<u>Remark.</u> Condition (4) is trivially satisfied in case (Λ_n) is increasing (the only situation considered, e.g., in Golomb [2]) since then $\mathrm{incl}(\ker T, \ker \Lambda_n)$ is decreasing as n increases. (4) is also satisfied in case $\underline{\lim}\, \Lambda_n \supseteq \ker T$. For, if $\kappa_\infty = 1$, there would exist, using the fact that $\dim \ker T < \infty$, an x in $\ker T$ and y_n in $\ker \Lambda_n$, all n, so that

$$\overline{\lim} \frac{(x, y_n)}{\|x\| \|y_n\|} = 1 .$$

But then, for all z_n in Λ_n,

$$\overline{\lim} \frac{\|x - z_n\|}{\|x\|} \geq \overline{\lim} \frac{|(x - z_n, y_n)|}{\|x\| \|y_n\|} = \overline{\lim} \frac{|(x, y_n)|}{\|x\| \|y_n\|} = 1$$

showing that x would not be in $\underline{\lim}\, \Lambda_n$. In particular, Lemma 2 follows.

Shekhtman finishes the proof of Theorem 1 with the following nice observation: Since (p_n) is bounded, so is (p_n^*), and, since $\underline{\lim}\, \Lambda_n = X$, by assumption, it follows that $p_n^* \xrightarrow{s} 1$. Consequently, $p_n \xrightarrow{w} 1$. But then $Tp_n \xrightarrow{w} T$, therefore $\|Tx\| \leq \underline{\lim} \|Tp_n x\|$, while also $\|Tp_n x\| \leq \|Tx\|$. Therefore $\|Tp_n x\| \longrightarrow \|Tx\|$, and so $Tp_n \xrightarrow{s} T$. It follows that

$$Qp_n = (T|_{\mathrm{ran} Q})^{-1} Tp_n \xrightarrow{s} (T|_{\mathrm{ran} Q})^{-1} T = Q$$

while, by the finite dimensionality of $\ker T = \mathrm{ran}(1-Q)$, $p_n \xrightarrow{w} 1$ implies $(1-Q)p_n \xrightarrow{s} 1-Q$. |||

Since ran $p_n^* = \Lambda_n$ while $\|p_n^*\| = \|p_n\|$, Shekhtman's argument shows that, for the particular sequence (p_n) of spline projectors,

$$p_n^* \xrightarrow{s} 1 \quad \text{implies} \quad p_n \xrightarrow{s} 1 \ .$$

Such an implication does not hold for general sequences of linear projectors, so that the converse of Theorem 1, if true, would again have to be proved using some special properties of the spline projectors. As it turns out, though, the converse does not hold even for spline projectors, as the following simple example shows.

Example. Take $X = Z = \ell_2$, $T = Q$, $1 - Q = P_{\mathrm{span}\{e_1\}}$, with $e_j := (\delta_{ij})_{i=1}^{\infty}$ and

$$\Lambda_n = \mathrm{span}\{e_2,\dots,e_{n-1},e_n+e_1\} \ .$$

Then $p_n x = \Sigma_{j<n} x(j)e_j + x(n)e_1$ which converges in norm to x since $\lim x(n) = 0$. In other words, $p_n \xrightarrow{s} 1$. On the other hand,

$$\mathrm{dist}(e_1, \Lambda_n) = \mathrm{dist}(e_1, \mathrm{span}\{e_1+e_n\}) = 1/\sqrt{2}$$

i.e., $e_1 \notin \underline{\lim}\ \Lambda_n$.

In this example, $\underline{\lim}\ \Lambda_n = \mathrm{span}\{e_2,e_3,\dots\} = (\ker T)\perp$, hence

(8) $$\underline{\lim}\ \Lambda_n \supseteq (\ker T)\perp \ .$$

We will show below that condition (8) is necessary for $p_n \xrightarrow{s} 1$. The example then also shows that $\underline{\lim}\ \Lambda_n$ need not contain anything else.

Proposition 2. Suppose that $p_n \xrightarrow{s} 1$. Then $\underline{\lim}\ \Lambda_n = X$ if and only if there exists a linear projector R with ran $R = \ker T$ which is the uniform limit of a sequence (R_n) of linear projectors with ran $R_n = \ker T$ and ran $R_n^* \subseteq \Lambda_n$, all $n \geq n_0$.

Proof. If $\underline{\lim}\ \Lambda_n = X$, then any bounded linear projector R on X with range $\ker T$ can be written

$$R \;=\; \sum_{i=1}^{r} x_i \otimes \lambda_i$$

for some basis $(x_i)_1^r$ of $\ker T$ and some dual set $(\lambda_i)_1^r$ of linear functionals. But, since $\underline{\lim}\, \Lambda_n = X$, we can find sequences $(\lambda_i^{(n)})$ with $\lambda_i^{(n)} \in \Lambda_n$, all n, and $\|\lambda_i - \lambda_i^{(n)}\| \longrightarrow 0,$, $i=1,\dots,r$. Since $\lambda_i x_j = \delta_{ij}$, all i,j, it is then also possible for all large enough n to find a basis $(x_i^{(n)})$ for $\ker T$ with $\lambda_i^{(n)} x_j^{(n)} = \delta_{ij}$ and then, necessarily, also $\|x_i - x_i^{(n)}\| \xrightarrow[n\to\infty]{} 0$. But then

$$R_n \;:=\; \sum_{i=1}^{r} x_i^{(n)} \otimes \lambda_i^{(n)}$$

converges in norm to R.

For the converse, if R_n converges in norm to R, then the sequence (S_n) given by

$$S_n \;:=\; R_n^* R_n \;+\; T^* T$$

converges in norm to

$$S \;:=\; R^* R \;+\; T^* T$$

The linear map S is selfadjoint, bounded, and is bounded below. Explicitly,

$$(Sx,x) \;=\; \|Rx\|^2 \;+\; \|Tx\|^2$$

while $TRx = 0$, hence

$$\|Tx\|^2 = \|T(1-R)x\|^2 \;\;\in\;\; \{\|(T|_{\mathrm{ran}(1-R)})^{-1}\|,\; \|T\|\}^2 \|(1-R)x\|^2 \;.$$

This shows that

$$(Sx,x) \;\;\in\;\; \{\min\{1, \|(T|_{\mathrm{ran}(1-R)})^{-1}\|\},\; \max\{1, \|T\|\}\}^2 (\|Rx\|^2 + \|(1-R)x\|^2)$$

while

$$\|Rx\|^2 \;+\; \|(1-R)x\|^2 \;\;\in\;\; \{1/2,\; 1 + 2\|R\|\|1-R\|\}\|x\|^2 \;.$$

We conclude that the bilinear form

$$(x,y)_S \;:=\; (Sx,y)$$

is an equivalent inner product on X and S is, therefore, in particular invertible. Since $S_n \longrightarrow S$ in norm, it follows that also S_n^{-1} exists for n sufficiently large and converges in norm to S^{-1}.

We now conclude from $p_n \xrightarrow{s} 1$ that also $S_n p_n S_n^{-1} \xrightarrow{s} 1$. In particular, for $x \in X$, setting $z_n := S_n^{-1} x$, we get

$$x \underset{\infty \leftarrow n}{\longleftarrow} S_n p_n z_n = R_n^* R_n p_n z_n + T^* T p_n z_n .$$

By construction, $\operatorname{ran} R_n^* \subseteq \Lambda_n$, while $T^* T p_n[X] \subseteq \Lambda_n$ due to the fact that (e.g. by (7)) $Tp_n = (1 - P_{T[\ker \Lambda_n]})T$, hence $Tp_n[X] \subseteq T[\ker \Lambda_n]\perp$ and so $T^* T p_n[X] \subseteq (\ker \Lambda_n)\perp = \Lambda_n$. But this shows that $x \in \underline{\lim}\, \Lambda_n$. |||

The last argument, carried out with $R_n = 1 - Q$, all n (recall that $Q = P_{(\ker T)\perp}$) shows that, for all $x \in X$,

$$(1-Q) p_n z_n + T^* T p_n z_n \underset{n \to \infty}{\longrightarrow} x .$$

But, since $\operatorname{ran} T^* \subseteq (\ker T)\perp = \operatorname{ran} Q$, this implies that $T^* T p_n z_n \longrightarrow Qx$ and so shows that $(\ker T)\perp = \operatorname{ran} Q \subseteq \underline{\lim}\, \Lambda_n$. This proves

<u>Corollary.</u> If $p_n \xrightarrow{s} 1$, then $(\ker T)\perp \subseteq \underline{\lim}\, \Lambda_n$.

<u>Theorem 2.</u> $p_n \xrightarrow{s} 1$ iff $\sup \operatorname{incl}(\ker T, \ker \Lambda_n) < 1$ and $(\ker T)\perp \subseteq \underline{\lim}\, \Lambda_n$.

<u>Proof.</u> Proposition 1 and the corollary to Proposition 2 show that the stated conditions are necessary for $p_n \xrightarrow{s} 1$. In order to show the sufficiency of these conditions, we need, by Proposition 1, only prove the following

<u>Proposition 3.</u> If (p_n) is bounded and $(\ker T)\perp \subseteq \underline{\lim}\, \Lambda_n$, then $p_n \xrightarrow{s} 1$.

<u>Proof.</u> Since $\ker T \subseteq \operatorname{ran} p_n$ and (p_n) is bounded by assumption, we are done once we show that $(\ker T)\perp \subseteq \underline{\lim}\, \operatorname{ran} p_n$. For this, let $z \in (\ker T)\perp = \operatorname{ran} Q$, and consider $y := T^* T z$, also in $\operatorname{ran} Q$. By assumption, $y = \lim y_n$, with $y_n \in \Lambda_n$, all n. Consequently,

$$\lim Q y_n = T^* T z \quad \text{and} \quad \lim (1-Q) y_n = 0 .$$

Now consider the bounded and boundedly invertible linear map

$$S := 1-Q + T^* T$$

on X introduced earlier for the proof of the corollary to Proposition 2. Note that $\ker T$ and $(\ker T)^{\perp} = \operatorname{ran} Q$ are both invariant under S, and $S = 1$ on $\ker T$. Hence we can write y_n as

$$y_n = (1-Q)y_n + T^*Tz_n$$

for some $z_n \in \operatorname{ran} Q$ and, since $y_n \longrightarrow y$, we have $z_n \longrightarrow z$. Further, for all $x \in \ker \Lambda_n$,

$$0 = (x,y_n) = (x,(1-Q)y_n) + (x,T^*Tz_n),$$

therefore

$$|(Tx,Tz_n)| \leq \|x\|\,\|(1-Q)y_n\| .$$

But this says (with (7)) that

$$\|z_n - p_n z_n\| = \|P_{T[\ker\Lambda_n]}Tz_n\| = \sup_{x\in\ker\Lambda_n} \frac{|(Tx,Tz_n)|}{\|Tx\|} \leq \|(T|_{\ker\Lambda_n})^{-1}\|\,\|(1-Q)y_n\| \xrightarrow[n\to\infty]{} 0$$

since, by the proof of Proposition 1 and the boundedness of (p_n), $\sup_n \|(T|_{\ker\Lambda_n})^{-1}\| < \infty$, while $\|(1-Q)y_n\| \longrightarrow 0$ as noted earlier. We conclude that $z = \lim z_n = \lim p_n z_n$. |||

Remark. In effect, the proof of propositions 2 and 3 relies on the fact that T^*T maps $\operatorname{ran} p_n \cap \operatorname{ran} Q$ and $\operatorname{ran} p_n^* \cap \operatorname{ran} Q$ onto each other.

Remark. As mentioned earlier, we have followed Shekhtman in making the assumption that $\dim \ker T < \infty$. But, our proof of Theorem 2 does not use that assumption. Theorem 2 is therefore true under the weaker assumption that p_n is defined for all n, which is assured in case $(\ker T) + (\ker \Lambda_n)$ is closed for all n, i.e., in case $\operatorname{incl}(\ker T, \ker \Lambda_n) < 1$, all n.

REFERENCES

[1] M.Atteia, Generalization de la definition et des proprietes des "spline-fonctions", C.R.Acad.Sci.Paris 260(1965),3550-3553.

[2] M.Golomb, Splines, n-widths and optimal approximations, MRC Tech.Report #784, Sep.1967, Madison,WI.

[3] B.Shekhtman, Unconditional convergence of abstract splines, J.Approximation Theory to appear.

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